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When are adaptive expectations rational? A generalization

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ABSTRACT

This note presents a simple generalization of the adaptive expectations mechanism in which the learning parameter is time variant. Expectations generated in this way minimize mean squared forecast errors for any linear state space model.

1. Introduction

Although it is tempting to see a dichotomy in the macroeconomics literature between those (early) models based on adaptive expectations and those (more recent) models based on rational expectations, the connection between the two mechanisms in fact runs deep. Indeed, the original contribution of Muth (1960) was to highlight that adaptive expectations are only rational in the sense of minimizing mean squared forecast errors under strict assumptions as to the underlying data generating process. This paper extends that insight to a more general case, and shows that for a very broad class of time series models – all those that can be written in linear state space form – a generalized form of adaptive expectations is rational in the sense of producing minimum mean squared forecast errors. The necessary generalization to the adaptive expectations mechanism is the introduction of a time-varying adaptation or learning parameter, which depends on the underlying characteristics of the model.

In addition to Muth (1960), who showed that adaptive expectations are rational if the data generating process is a random walk with noise, contributions by Theil and Wage (1964) and Nerlove and Wage (1964) addressed the optimality of the closely related procedure of exponential smoothing. All three papers are special cases of the more general approach taken here, which uses the Kalman Filter to derive similar results for the full set of time series models that can be written in linear state space form. The only previous research that uses the Kalman Filter in this way is Cuthbertson (1988), who also focuses on an adaptive expectations model with a time-varying adjustment parameter. However, he does not provide a general framework that establishes the optimality of such forecasts, but instead relies on a series of special cases. In addition, Farmer (2002) develops a generalized version of adaptive expectations which he shows to be rational under given circumstances, but his approach again relies on more specific cases than the one used here. The present contribution represents a further generalization of both approaches.

The paper proceeds as follows. To introduce the material, Section 2 provides an alternative proof of the proposition in Muth (1960) by transforming the model into linear state space form and applying the Kalman Filter. Section 3 presents the general problem for any linear state space model, applies the Kalman Filter, and shows that its forecasts can be expressed as a generalization of the traditional adaptive expectations model. Section 4 concludes.

2. Adaptive expectations, rationality, and the Kalman filter

As is well known, the traditional adaptive expectations model applied to, for example, a commodity price p_t, takes the following form:

$$p_t^* = p_{t-1}^* + \beta \left(p_{t-1} - p_{t-1}^*\right) \quad (1)$$

where stars indicate expected prices, and $0 < \beta < 1$ is a learning parameter that determines the speed with which prior errors are

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"corrected" when making forecasts. Early work using adaptive expectations justified it on intuitive grounds (e.g., Nerlove, 1956). Muth (1960) subsequently showed that expectations formed in this way are rational in the sense of minimizing mean squared forecast errors provided that prices evolve according to a random walk, i.e.:

$$p_t = p_{t-1} + e_t \tag{2}$$

where e_t is a standard, white noise error term. Although the meaning of the term "rational expectations" has evolved in the more recent literature, I use it here in the same sense as Muth (1960), i.e. expectations are rational if they minimize mean-squared forecast errors.

By way of introduction to the generalized model presented in the next section, it is useful to provide an alternative proof of Muth's result. This is easily done using the Kalman Filter. To set up the problem, I rewrite the price process in terms of a stochastic trend μ_t, as follows:

$$p_t = \mu_t + \eta_t \tag{3}$$

$$\mu_t = \mu_{t-1} + \omega_t. \tag{4}$$

An agent who is rational produces one-step ahead forecasts (m_t) of p_t that minimize the mean squared forecast error. Since the system described by Eqs. (3) and (4) takes the form of a linear state space model, minimum mean squared error forecasts can be obtained recursively by applying the Kalman Filter[1]:

$$v_t = p_t - m_t \tag{5}$$

$$V_t = \text{var}(v_t) = Q_t + \sigma_\eta^2 \tag{6}$$

$$K_t = \frac{Q_t}{V_t} \tag{7}$$

$$m_{t+1} = m_t + K_t v_t \tag{8}$$

$$Q_{t+1} = Q_t\left(1 - K_t\right) + \sigma_\omega^2. \tag{9}$$

Substituting Eq. (5) into Eq. (8) gives:

$$m_{t+1} = m_t + K_t\left(p_t - m_t\right) \tag{10}$$

which takes the traditional adaptive expectations form of Eq. (1) provided that K_t is a constant. To prove that this is the case, substitute Eq. (6) into Eq. (7) to give:

$$K_t = \frac{Q_t}{Q_t + \sigma_\eta^2}. \tag{11}$$

Time invariance of K therefore reduces to time invariance of Q, i.e. $Q_{t+1} = Q_t = Q$. Substituting (11) into (9) and imposing the equality yields:

$$Q = Q\left(1 - \frac{Q}{Q + \sigma_\eta^2}\right) + \sigma_\omega^2. \tag{12}$$

Solving for Q and retaining only the positive solution because it is a variance gives:

$$Q = \frac{\sigma_\omega^2 + \sqrt{\sigma_\omega^4 + 4\sigma_\omega^2\sigma_\eta^2}}{2} \tag{13}$$

which must be strictly positive for any non-trivial model. It therefore follows that K_t is indeed constant, and that Eq. (10) is in the traditional adaptive expectations form. Moreover, it follows from Eq. (11) and the fact that σ_η^2 is strictly positive that $0 < K < 1$, as in the traditional model.

[1] Standard sources such as Durbin and Koopman (2001) and Harvey (1989) provide full derivations and proofs of the properties of the Kalman Filter.

3. Generalized adaptive expectations

This section extends the analysis in Section 2 to a more general setting. Specifically, I use the Kalman Filter to show that the generalized form of adaptive expectations given by Eq. (10) is rational for a broad range of data generating processes in a multivariate setting. The sense in which Eq. (10) represents a generalization of the adaptive expectations mechanism is that the learning parameter β is not time invariant, as in the original model, but instead can change over time.

The linear state space model takes the following general form, using matrix notation:

$$\mathbf{y_t} = \mathbf{Z_t}\alpha_\mathbf{t} + \varepsilon_\mathbf{t} \quad \varepsilon_\mathbf{t} \sim \mathbf{N}\left(\mathbf{0}, \mathbf{H_t}\right) \tag{14}$$

$$\alpha_\mathbf{t+1} = \mathbf{T_t}\alpha_\mathbf{t} + \mathbf{R_t}\eta_\mathbf{t} \quad \eta_\mathbf{t} \sim \left(\mathbf{0}, \mathbf{Q_t}\right) \tag{15}$$

$$\alpha_\mathbf{1} \sim \mathbf{N}\left(\mathbf{a_1}, \mathbf{P_1}\right). \tag{16}$$

It is a very general specification that includes, for example, all models in the ARIMA class. By defining $\mathbf{y_t}$ as a $p \times 1$ vector, it also includes multivariate extensions of the ARIMA class. In addition, appropriate specification of the matrices $\mathbf{Z_t}$ and $\mathbf{T_t}$ allows for the imposition of cross-equation restrictions consistent with an underlying model of the economy.

The Kalman Filter for the model in Eqs. (11) and (12) is given by:

$$\mathbf{v_t} = \mathbf{y_t} - \mathbf{Z_t a_t} \tag{17}$$

$$\mathbf{K_t} = \mathbf{T_t P_t Z_t' F_t^{-1}} \tag{18}$$

$$\mathbf{a_{t+1}} = \mathbf{T_t a_t} + \mathbf{K_t v_t} \tag{19}$$

$$\mathbf{F_t} = \mathbf{Z_t P_t Z_t'} + \mathbf{H_t} \tag{20}$$

$$\mathbf{L_t} = \mathbf{T_t} - \mathbf{K_t Z_t} \tag{21}$$

$$\mathbf{P_{t+1}} = \mathbf{T_t P_t L_t'} + \mathbf{R_t Q_t R_t'}. \tag{22}$$

Substituting Eq. (14) into Eq. (16) gives:

$$\mathbf{a_{t+1}} = \mathbf{T_t a_t} + \mathbf{K_t}\left(\mathbf{y_t} - \mathbf{Z_t a_t}\right) \tag{23}$$

and premultiplying by $\mathbf{Z_{t+1}}$ gives:

$$\mathbf{Z_{t+1} a_{t+1}} = \mathbf{Z_{t+1} T_t a_t} + \mathbf{Z_{t+1} K_t}\left(\mathbf{y_t} - \mathbf{Z_t a_t}\right). \tag{24}$$

To see that Eq. (21) takes the form of generalized adaptive expectations, note that from (12) and (13) $E\left[\mathbf{y_t}\right] = \mathbf{Z_t a_t} = \mathbf{Z_t T_{t-1} a_{t-1}}$, and thus:

$$E\left[\mathbf{y_{t+1}}\right] = E\left[\mathbf{y_t}\right] + \mathbf{Z_{t+1} K_t}\left(\mathbf{y_t} - E\left[\mathbf{y_t}\right]\right). \tag{25}$$

In general, $\mathbf{Z_{t+1} K_t}$ will be time-varying, and so the coefficient of adaptation will change over time, rather than remain constant as in the traditional adaptive expectations model.

4. Conclusion

This note has developed a simple generalization of the adaptive expectations mechanism in which the learning parameter is time-varying. Whereas standard adaptive expectations are only rational when the underlying data generating process is a random walk with noise, the generalization is rational for a much broader class of time series models. Because the proof of rationality relies on the Kalman Filter, generalized adaptive expectations can easily be seen to be rational for any time series model that can be written in linear state space form. This class of models is very broad, and includes, for example, all ARIMA models. The analysis presented here highlights the connection between adaptive and rational expectations, in an extension of the original work of Muth (1960).

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